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HISTORICAL DYNAMICS OF IMPLICIT AND INTUITIVE ELEMENTS OF MATHEMATICAL KNOWLEDGE

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The article deals with historical dynamics of implicit and intuitive elements of mathematical knowledge. The author describes historical dynamics of implicit and intuitive elements and discloses a historical and evolutionary mechanism of building up mathematical knowledge. Each requirement to increase the level of theoretical rigor in mathematics is historically realized as a three-stage process. The first stage considers some general conditions of valid mathematical knowledge recognized by the mathematical community. The second one reveals the level of theoretical rigor increasing, while the third one is characterized by explication of the hidden lemmas. A detailed discussion of historical substantiation of the basic algebra theorem is conducted according to the proposed technique.

Keywords: *history of mathematics, level of theoretical rigor, three-stage process, hidden lemmas, evolutionary mechanism of mathematical knowledge growth.*

Introduction

New mathematical knowledge obviously involves a great degree of intuition, that making it constantly look insufficiently rigid and, consequently, non-valid enough. Therefore, at a certain historical moment of mathematical development there inevitably appears a task of increasing the level of theoretical rigor, within which the validity of mathematical knowledge must be confirmed again. This means that rigid mathematical knowledge has to be verified with a greater extent of accuracy. However, no one doubts the fact that this problem can be solved and, thus, the validity of the earlier proven mathematical knowledge can be supported, and this knowledge can be substantiated in accordance with a higher degree of rigidity.

The increase of the theoretical rigor level in history seems to be largely possible due to the growth of mathematical knowledge when its implicit intuitive element is made explicit. It means that in order to discover the mechanism of this growth historical dynamics of the implicit intuitive element of mathematical knowledge should be primarily investigated. It may be stated that a historical and evolutionary mechanism of mathematical knowledge growth, which is to be made explicit later, should express that dynamics in some way. Historically, mathematical knowledge growth is known to result from mathematical theories turned into axioms and mathematical methods developed into algorithms. That implies that the required growth can be achieved only at the expense of explicating the implicit intuitive element of the mathematical theory.

Before one may be able to handle our main objective, i.e. to explicate a historic and evolutionary mechanism of mathematical knowledge growth, it is necessary to disclose in what form the implicit intuitive element is contained in the mathematical theory. It stands to reason that a valid mathematical theory is characterized by harmonic nature, although the "price" of this harmony is extremely high. The implicit intuitive element is generally incorporated into the mathematical theory basically in the form of groundless statements, in other words, in the form of hidden lemmas. As a rule, such implicit grounds are intersubjective within a concrete social and cultural period, and, as such, can be explicated only in case the social and cultural "vision" of mathematicians is changed, i.e. only when mathematicians with a non-standard way of thinking emerge. For instance, implicit geometry prerequisite, which later became known as Jordan's lemma, and which was in a due time a basic one for Euclid, was revealed only in the 19th century by the English mathematician and writer L. Carrol [1, 51]. In addition, Cauchy used such a hidden lemma in the theorem about polyhedrons. The essence of it lies in the fact that all polyhedrons were considered as simple ones. The mathematician Beckker, who was the first to find it in the Cauchy's paper, even called it "an error" [1, 66]. But it is a typical example of an implicit prerequisite in the mathematical theory.

Results and Discussion

It can be stated that each historic period generates specific implicit prerequisites of social and cultural nature, involving foremost notions of mathematical methods, a level of theoretical rigor, and those of the mathematic knowledge ideal. These implicit prerequisites can be considered as a common element of thinking of the mathematical community members at a certain historical period, therefore they are intersubjective and implicit only within the above period and later become accessible for revealing [2]. In this connection it may be admitted that axioms first found out by Euclid, had been used by ancient mathematicians as grounds for geometrical reasoning at an implicit level. It may be suggested that the above discovery was made possible due to the ancient idea of the out-of-empiric origin of knowledge. According to it, any knowledge is gained by speculation and is a direct result of thinking. The Plato's idea that knowledge is something implicitly contained in thinking, underlies such an idea of getting knowledge.

The reason for such an explication is just further increase in the level of theoretical rigor for a developed mathematical theory. This is by all means accompanied by discovering some implicit prerequisites, leading to a notable growth of the theoretical rigor level in mathematics, and consequently, to strengthening the substantiating basis for not only mathematical theories alone, but mathematical knowledge in general. The substantiating basis is understood as an aggregate of all explicit and grounded prerequisites applicable in the mathematical theory. All unsubstantiated implicit or explicit prerequisites, i.e. those of intuitive nature, do not constitute the substantiating basis. It is clear that the substantiating basis of a concrete mathematical theory must develop, that is the theoretical rigor degree in mathematics increases so does its power. For instance, the substantiating basis of the theorem of polyhedrons proved by Cauchy could not include a hidden lemma on simple polyhedrons, whereas this lemma was already contained in this theorem proof explicated later by the mathematician Beckker [1, 66]. It is evident that since Beckker's paper appeared the power of the substantiating basis of the polyhedron theorem has risen. Similarly, the power of the substantiating basis of mathematics as a whole increased either.

What is the mechanism of that process? What is the relationship between the degree of the theoretical rigor and the definite "amount" of implicit prerequisites, and the certain level of implicit knowledge in the mathematical theory? To put it more precisely, how the growth of the substantiating basis of a mathematical theory, i.e. the growth of new knowledge in mathematics goes. To answer this question the following three-stage scheme is demonstrated.

Let us assume that at the first stage some concrete mathematical theory *T* to a particular degree corresponds to the theoretical rigor *U* with the validity *N* at a certain historical moment *t.* That means that the theory *T* is proved and, thus, is consistent. In parallel, from the point of view of the historic perspective of a likely rigor increase within the framework of the theoretical rigor *U*, it can be represented by a combination of explicit mathematical statements *P* and a set of implicit prerequisites *PN*. However, as a concrete mathematical theory is being considered, and is recognized true (consistent) by the mathematical community, it is necessary to set *PN* to zero*.* This means that implicit knowledge undoubtedly constituting the prerequisites of this theory *T* is absolutely unrealizable. It influences the formation of the mathematical contents proper, but it is not substantiated and formulated anywhere, and is not realized by the mathematical community as something mathematically important. Such implicit prerequisites are at best verbalized as some non-rigid mathematical reasonings within the corresponding historical context, i.e. some heuristic considerations of semi-intuitive nature. Implicit prerequisites are inherent but unrealizable components of the mathematical theory *T*.

Then at a remote definite history moment $t = t + \Delta t$ a notable change in the degree of the theoretical rigor *U* is set in mathematics, with $U = U + \Delta U$. From the point of view of this increased level of the theoretical rigor *U* our theory *T* is seen insufficiently rigid from the mathematical point of view, since it is no longer possible set a *PN* element to zero. Let us represent it as a collection of the finite number of implicit prerequisites *PN*. These implicit prerequisites *PN* are thought to determine insufficient theoretical rigor of our theory *T* within higher degree of the theoretical rigor *U*. The rigidity of the mathematical proof is determined as the absence of such elements which could cause inconsistency in *T*. The situation of a defective theoretical rigor of the theory *T* obviously calls for the inconsistency to rise, resulting in the necessity of explicating intuitive elements, with the implicit prerequisites of the theory *T* inclusive. Explicating implicit prerequisites is possible only in case they are verbalized, otherwise their prediction is unlikely as demonstrated by the real mathematical history. Revealing latent theoretical prerequisites is as creative a process in mathematics as getting new knowledge. In mathematical reasoning implicit prerequisites greatly complicate substantiation as an evolutionary historic process of explicating an implicit intuitive element of the mathematical theory.

The only way to restore the reputation of the mathematical theory *T* as an absolutely valid one is to reveal all implicit prerequisites which were discovered within the increased degree of the theoretical rigor *U* and formed an aggregate of the element *PN* of the theory *T*. The mathematical theory *T* on the first stage of its historic substantiation was included to the general mathematical context, and now cannot be abandoned as causing doubt by its validity. Moreover, its consistency is accepted due to the increased degree of the theoretical rigor *U* either. A historical struggle for the rigor level of the theory T as a valid mathematical knowledge requires explication of all prerequisites verbalized within a new degree of the theoretical rigor *U*. This process contributes to the necessary growth of the mathematical knowledge content for the theory *T* taking place on the second stage of historic substantiation of the mathematical theory *T*.

Now that historical explication of implicit prerequisites within a new and higher degree of the theoretical rigor *U* having been implemented, the mathematical theory *T* confirmed its status as a valid one. In spite of the fact that implicit prerequisites in the theory *T* were successfully overcome contributing to the new knowledge growth, it should not be concluded that all implicit prerequisites constituting the theory *T* were discovered and included in the mathematical theory *T*. This theory has not been absolutely substantiated and its consistent character has not been fully confirmed either. Historic substantiation which in reality turns this theory *T* into axioms can be considered as theoretically unlimited repetition of triad cycles, the stages of which were disclosed in the above scheme. It should be taken into account that the *PN*-element of the theory *T*, containing all its implicit prerequisites also comprises its basic reasons. In the given scheme of substantiating the mathematical theory *T* they are not separated from the hidden lemmas in the theorem proofs, as they change nothing in the conception of the scheme, but deprive it of its clarity.

Due to the extreme awkwardness of a full historic substantiation, i.e. axiomising a concrete mathematical theory, to be submitted here such a historic substantiation in mathematics accompanied by explicating implicit prerequisites will be exemplified in historic specification of the basic algebra theorem, originally given by D'Alembert in 1746. Its formulation, considerably differing from the one accepted in modern mathematics, was initially given by H. Jirard and R. Decartes. It is D'Alembert who in "Investigations of Integral Calculus" formulates the basic algebra theorem as follows: any algebraic polynomial with real coefficients is expanded into the product of linear and square-law real factors. The proof by D'Alembert was of an analytical character. K. Gauss in his doctorate thesis of 1799 was quite fair to point to the lack of rigor in D'Alembert's argumentations, already unacceptable for the mathematics of the 19th century. In particular, D'Alembert did not prove the original assumption on the possibility of expanding the algebraic function in a convergent series, which in K. Gauss's opinion, definitely represented an implicit prerequisite [3]. Also some reasonings relating to the infinitesimal failed to be rigid in one of the corollaries. The essential remark by K. Gauss is the one that an algebraic function does not necessarily reach its bounds underlying one of the corollaries [3], and not belonging to implicit prerequisites.

Further refinement of the proof of the basic algebra theorem, given by K. Gauss was performed by F. Klein, who also explicated its implicit prerequisites. According to the reported idea of historic explicating of implicit prerequisites of the mathematical theory, F. Klein's refinement represents the next stage of this explication. F. Klein contends that K. Gauss in his proof, for instance, "uses here the properties of algebraic curves", in particular, the one stating that a curve cannot be interrupted [4]. Later on F. Klein stresses that this "fact is formulated but undergoes no further analysis" [4, 69]. That prerequisite employed by K. Gauss, is implicit as well, but unlike the previous one which is not only implicit theoretically but cannot be verbalized, the former had been verbalized by K. Gauss, being theoretically implicit though. Along with that, in K. Gauss's proof some fundamental theorems of continuity for two-dimensional regions stand to reason – for example, "the theorem that two intersecting curves are bound to have an intersection point" [4, 69].

Generally, according to F. Klein and common ideas of modern mathematics, K. Gauss's proof obviously lacks the substantiation of the fact of continuity of the function under consideration, which K. Gauss, evidently, uses as an implicit prerequisite. Thus it is necessary to formulate the assumption of the theory of real numbers, the fullness of which is achieved due to Dedekind's cuts [4, 70–71]. In conclusion of his refinement to K. Gauss' proof of the basic algebra theorem F. Klein makes a remark that all above was overcome only in 1817 in B. Boltzano's work "Purely analytical proof of the theorem that between two values there is at least one real root of the equation yielding the results of the opposite signs", in which B. Boltzano explicated the indicated implicit prerequisite in K. Gauss's proof [4, 70–71]. F. Klein's refinements certainly were performed much later; the order of refinements is not of principal value here. It is important that no one of the mathematicians explicating the proof of the basic algebra theorem by D'Alembert questioned the consistency of the starting assumptions of this proof, its validity and the possibility of its further sound explication. It is evident that each stage of explicating D'Alembert's proof of the basic algebra theorem performed by K. Gauss, B. Boltzano and F. Klein was gradually collecting new knowledge quite agreed with the presented idea of historic substantiation of the mathematical theory.

A historic and mathematical analysis allows us to conclude that the historic substantiation of many mathematical assumptions is implemented similarly. A real historic process of substantiation in mathematics proceeds in a more complex way. Concrete implicit prerequisites of the mathematical theory can be discovered and verbalized, or cannot be discovered either, since the process of explicating implicit prerequisites is unpredictable in any certain case. Though in the mathematical science theory, a general historic trend to explicate implicit prerequisites undoubtedly takes place. Yet it should be borne in mind that the conditions for implicit prerequisites to be explicated are far from frequent, since the former needs changes in a theoretically implicit paradigm and a social and cultural layer of mathematicians' thinking. In other words, it is necessary for mathematicians' "vision" to modify.

This phenomenon is thought to be the main reason for impossibility to ensure that the increase in the degree of theoretical rigor in mathematics will result in some concrete implicit prerequisites being certainly explicated. That means that for any concrete theory or proposition it cannot be guaranteed that further increase in the degree of theoretical rigor necessarily implies explicating the very theory or proposition. Even in the case of finite number of implicit prerequisites there is no confidence that in some certain moment of the real history these very implicit prerequisites will be discovered. Concrete implicit prerequisites are disclosed by a concrete subject, and at this level the process of explication is intuitive and, consequently, irrational. It can be deduced that in theory it cannot be guaranteed that the process of profound historic substantiation for some certain mathematical theory T can be limited in time and, as a result, completed [4, 67]. As for the present context that means that in theory the repetition of triad cycles in the process of historically explicating implicit prerequisites of a mathematical theory is not generally finite. Here only historic and evolutionary substantiation of mathematical knowledge is implied, that being realized within mathematical thinking of an axiom type. Just the explication of some part of all implicit prerequisites with each increase in the degree of theoretical rigor in mathematics is possible and can be ensured historically.

Conclusion

1. In conclusion, we note that the historical and evolutionary mechanism of mathematical knowledge growth, the essence of which actually consists in a multi-step explication of implicit premises in theorem proofs, resonates in some moments with the theory of mathematical concepts by I. Lakatos. Counter-examples accessed randomly are considered a driving force for explicating mathematical proofs. It seems, however, that the sense of counter-examples predominantly lies in discovering implicit premises in proofs. However, we note that the explication of hidden lemmas in mathematical proofs is under the influence of counterexamples, which gives I. Lakatos reason to doubt the status of deductive mathematics. But the example of explicating implicit premises of the fundamental algebra theorems shows that counter-examples are not always necessary in detecting hidden lemmas of mathematical proofs. The main factor is the mathematicians' notion about the

lack of rigor in mathematical knowledge and their confidence that this level can be enhanced by clarification of mathematical proofs. In fact, unstated assumptions, such as in the proof of the fundamental theorem of algebra, were detected by different mathematicians at different times, solely as a result of their desire to make this proof more stringent. None of the mathematicians even mentioned any counter examples necessary for clarifying the evidence. Thus, I. Lakatos' conclusions undermining the status of deductive mathematics are quite private and cannot be extended to the whole mathematical science.

2. Note that our task was not to criticize Lakatos at all. This criticism is not important in itself, it simply allows you to better understand the idea of the author. Besides, I think that Lakatos should be criticized, because it was his work on the history of mathematics that had an enormous influence on the philosophy of mathematics and gave an impetus to the development of mathematical empiricism. We are unable to move forward without highlighting the key points in the established scientific context. It should also be noted that the value of mathematics is determined primarily by the deductive nature of its theories and its priori grounds. It seems that the reference to specific events in mathematical history is a good way to defend mathematics from attacks and insinuations in the spirit of empiricism. As for the implicit assumptions found in mathematical proofs, it appears that this does not undermine traditional status of deductive mathematics, and must be regarded as an inherent feature of mathematical thinking. The history proves that almost all the unstated assumptions are explicated over time. The number of the remaining is so small that does not affect the validity of mathematical theories, except situations when you need to appeal to actual infinity. But this question must be considered separately [See 5, 160–168].

3. In general, the approach basing on the idea of tacit knowledge ontologically reveals a true history of a mathematical reasoning mechanism, consisting in the explication of various kinds of implicit assumptions. The main factor of implementing such an explication is a critical epistemological setting of the mathematical community, typical to the leading mathematicians.

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